Currency Risk Modelling by GARCH-Copula Model

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Abstract
Time series modelling and subsequent risk estimation is difficult but important activity of any financial institution. Financial time series are characterized by volatility clustering and heavy-tailed distributions of returns. Both these characteristics have a great influence for risk estimation. Especially when modelling more-dimensional probability distribution, the shocks in terms of extreme losses (or returns) in particular risk drivers are usually more correlated than the losses (returns) closer to the mean. In this paper we focus on the GARCH-copula models. The copula functions are the tool which allows us to model the dependence among individual risk drivers. Student probability distribution and various copula functions are assumed in the paper. On the other hand, GARCH model allows depicting the volatility clustering. These joined models are backtested on chosen dataset and the exceptions (i.e. their quantity and distribution in time) are statistically tested by Kupiec and Christoffersen tests.

Keywords: currency risk, copula function, GARCH model, backtesting
JEL codes: C53, F31, G17

1. Introduction

Modelling of the currency risk is important part of risk estimation and management of all financial institutions. These institutions measure risk very carefully, as the future unexpected losses can have serious consequences both for the management and share-holders. The financial risk (which is the umbrella term for all kinds of risk financial institution face to) can be divided into credit risk, liquidity risk, operational risk and market risk. There are also other (special) types of risks (e.g. reputation risk), however these four risk categories are the most important ones. The paper is focused on market risk, i.e. the risk of losses in positions arising from the movements in market prices. Depending on the type of risk factor the following risks can be distinguished: interest rate risk, currency risk, equity risk and commodity risk. To estimate these risks soundly the accurate model of financial time series evolution has to be applied. In this paper we focus on currency risk, however the proposed methodology can be applied to any other component of market risk.

When modelling financial time series we have to deal with the following issues. Firstly, the empirically observed returns of financial time series are characterized by fatter tails compared to the Gaussian (normal) distribution. Thus, it can be concluded, in line with Fama (1965), that Gaussian distribution is not appropriate for modelling of financial returns. Next, the volatility of returns is not constant over time, but is rather clustered. Thus, for the same asset, the periods with high volatility (high gains/losses) can be seen as well as the periods in which volatility is low (the gains/losses are very small). This issue can be tackled by the volatility modelling. The first volatility model, ARCH, were proposed by Engle (1982) and then expanded by Bollerslev (1986) to GARCH model. At present, there are many modification to this original models, e.g. the asymmetrical one proposed by Glosten, Jagannathan and Runkle (1993). However there is a common part (and logic) of these modified models, that is similar to the original GARCH model. The last issue to deal with is the dependency among the particular time series. Generally the returns are not strongly correlated when they are close to zero, however in the tails the correlation increases. Appropriate tool for dependency modelling are copula functions based on Sklar's theorem (Sklar, 1973), which allows to decompose the joint distribution into marginal distributions and copula function. The distribution of particular time series is then modelled by marginal distributions, while dependency is tackled only by copula function.

Recently, there have been published several papers dealing with the analysis of risk models via backtesting procedure. For instance, models assuming conditional volatility were backtested on foreign exchange time series by Alexander and Sheedy (2008) or on stock market index S&P 500 by Kresta (2013). Similar multi-position models, however neglecting the conditionality of volatility, were analysed e.g. by Rank (2007) or Kresta and Tichý (2012a, 2012b). Joint GARCH-copula models with the application to portfolio of indices were discussed by Huang et al. (2009) and for foreign exchange sensitive portfolio by Wang at al. (2010). The joint GARCH-copula model applied in this paper is similar to the ones applied before, however in this paper the structure of the GARCH model and type of copula function are not fixed, rather they are chosen in each backtesting step.

The goal of the paper is to propose methodology for accompanying the GARCH model, as proposed by Bollerslev, with copula functions and then backtest this proposed GARCH-copula model on foreign exchange time series for Value at Risk estimation.

The paper is organized as follows. First, the GARCH-copula model is described in the next section, including both the GARCH model and copula functions description. In the third section the backtesting procedure is briefly explained. The last section is application, as the results of proposed GARCH-copula model are presented within.

2. GARCH-copula models

In this paper we try to examine whether the GARCH models with residuals joined by a suitable copula functions are eligible to estimate the currency risk of foreign exchange rate sensitive portfolio. We assume several risk factors, which are modelled by GARCH model (described in subsection 2.1). The residuals (i.e. the random terms) in this model are modelled by copula functions (introduced in subsection 2.2).

To model the future evolution of financial time series the following procedure should be undertaken. First, parameters of GARCH model are estimated for each particular risk driver from past observations. When GARCH models are estimated, the residuals (observed in past) can be obtained. These are put together into a matrix and parameters of copula function are then estimated. By this way all the necessary parameters are estimated. For the simulation the sequence is opposite. First, random residuals are simulated, while the dependency among them is maintained by means of the estimated copula function. Then, these simulated residuals are transformed to the return time series by means of estimated GARCH models. These returns can be then easily utilized for computation of expected portfolio return or its risk.

2.1 Volatility models

Volatility models have become important tool in time series analysis, particularly in financial applications. Engle (1982) observed that, although the future value of many financial time series is unpredictable, there is a clustering in volatility. He proposed autoregressive conditional heteroskedasticity (ARCH) process, which has been later expanded to generalized autoregressive conditional heteroskedasticity (GARCH) model by Bollerslev (1986). There are also other volatility models such as GJR, IGARCH, FIGARCH, GARCH-M, EGARCH, etc. For their description see e.g. (Arlt and Arltová, 2007).

For time series modelling the conditional mean can be assumed, i.e. time series $\{x_t\}_{t=1}^{N}$ is modelled as follows,

$$x_t = \mu_0 + \sum_{i=1}^{R} \mu_i \cdot x_{t-i} + \sigma_t \cdot \tilde{\varepsilon}_t , \tag{1}$$

$$\tilde{\varepsilon}_t \sim t_\nu (0,1), \tag{2}$$

where μ_0 is unconditional mean of the series, μ_i are autocorrelation coefficients for lag 1 up to R, σ_t is modelled standard deviation (volatility) by the GARCH model (1) and $\tilde{\varepsilon}_t$ is a random number from

Student probability distribution (henceforth standardized residual or residuals). In the paper Student distribution (henceforth t-distribution) is assumed. The student distribution is chosen for its ability to model heavier tails (higher kurtosis of probability distribution), which are usually present in financial time series of returns.

The GARCH model (Bollerslev, 1986) was proposed as the extension of ARCH model in order to avoid problematic parameters estimation, when there are many of them. The model takes the following form,

$$\sigma_t^2 = \alpha_0 + \sum_{i=1}^{P} \alpha_i \cdot \sigma_{t-i}^2 + \sum_{j=1}^{Q} \beta_j \cdot \varepsilon_{t-j}^2 , \tag{3}$$

where α_0, α_i and β_j are parameters needed to be estimated. The positive variance is assured if $\alpha_0 > 0$, $\alpha_i \geq 0 \; \forall i$ and $\beta_j \geq 0 \; \forall j$. The model is stationary if $\sum_{i=1}^{P} \alpha_i + \sum_{j=1}^{Q} \beta_j < 1$.

It is usually problematic to choose the correct values of lags (*R*, *P*, *Q*). In this paper, in order to obtain statistically valid model, we proceed as follows:

- The full model as specified by the equations (1) and (3) is estimated with *R*, *P*, *Q* equal to 2.
- The statistical significance of parameter μ_2 is tested (t-test). If the parameter is found statistically not significant it is left out from the model and new model is estimated (*R*=1*)*. In the case of new specification of the model, the parameter μ_1 is tested in the same way.
- The statistical significance of parameter μ_0 is tested. If not found significant it is left out from the model and new model is estimated.
- Parameters β_2 , α_2 , β_1, α_1 (in this order) are gradually statistically tested. If they are found statistically insignificant the values of *P* (*Q*) are decreased to one or to zero. However, due to the estimation procedure requirements, *Q* is set to zero only if *P* is set to zero.

Applying this procedure we obtain the model, in which all the estimated parameters are statistically different from zero (the parameter α_0 is not statistically tested as it is required to be present in the model, in order to have positive variance).

2.2 Copula functions

When generating standardized residuals in GARCH model, the mutual dependence has to be considered. A useful tool for dependence modelling are the copula functions, i.e. the projection of the dependency among particular distribution functions into $[0,1]$,

$$C : [0,1]^n \to [0,1] \text{ on } R^n, \, n \in \{2,3,\ldots\}. \tag{4}$$

Basic reference for the theory of copula functions is Nelsen (2006), while Rank (2007) and Cherubini et al. (2004) target mainly on the application issues in finance.

Actually, any copula function can be regarded as a multidimensional distribution function with marginals in the form of standardized uniform distribution. For simplicity, assume two potentially dependent random variables with marginal distribution functions F_X , F_Y and joint distribution function $F_{X,Y}$. Then, following the Sklar's theorem (Sklar, 1959):

$$F_{X,Y}(x, y) = C(F_X(x), F_Y(y)). \tag{5}$$

If both F_X , F_Y are continuous, a copula function C is unique. Sklar's theorem implies also an inverse relation,

$$C(u, v) = F_{X,Y}\left(F_X^{-1}(u), F_Y^{-1}(v)\right). \tag{6}$$

The formulation above should be understood such that the joint distribution function gives us two distinct information: (i) marginal distributions of random variables, (ii) dependency function of distributions. Hence, while the former is given by $F_X(x)$ and $F_Y(y)$, a copula function specifies the dependency, nothing less, nothing more. That is, only when we put both information together, we have sufficient knowledge about the pair of random variables X , Y .

2.2.1 Elliptical and Archimedean copula functions

With some simplicity we can distinguish the elliptical copula functions and Archimedean copula functions. The elliptical copula functions are based on some elliptical joint distribution, such as Gaussian copula function based on joint Gaussian distribution,

$$C_R^{Ga}(u,v)=\int_{-\infty}^{\Phi^{-1}(u)}\int_{-\infty}^{\Phi^{-1}(v)}\frac{1}{2\pi\sqrt{1-R^2}}e^{\left[\frac{2Rst-s^2-t^2}{2-2R^2}\right]}dsdt\,, \tag{7}$$

where R is the correlation coefficient, or Student copula function based on the Student t distribution,

$$C_{R,\upsilon}^{St}(u,v)=\int_{-\infty}^{t_\upsilon^{-1}(u)}\int_{-\infty}^{t_\upsilon^{-1}(v)}\frac{1}{2\pi\sqrt{1-R^2}}\left[1+\frac{s^2+t^2-2Rst}{v\left(1-R^2\right)}\right]^{-\frac{\nu+2}{2}}dsdt\,, \tag{8}$$

where R is again the correlation coefficient and ν stands for degrees of freedom of the Student t distribution.

On the other hand, Archimedean copula functions are defined on the basis of function ϕ called generator. Generator is continuous, decreasing and convex function such that $\phi(1)=0$ and for a strict generator also stands that $\phi(0)=+\infty$. Archimedean copula functions can then be defined as follows,

$$C_{\phi,\phi^{[-1]}}^{Arch}(u,v)=\phi^{[-1]}(\phi(u),\phi(v))\,, \tag{9}$$

where $\phi^{[-1]}$ is the pseudo-inverse function such that $\phi^{[-1]}(\phi(v))=v$ for every $v\in[0;1]$. The most known Archimedean copula functions are: Gumbel copula function (Gumbel, 1960),

$$C_a^{Gl}(u,v)=\exp\left\{-\left[(-\ln u)^a+(-\ln v)^a\right]^{\frac{1}{a}}\right\}, \tag{10}$$

Clayton copula function (Clayton, 1978),

$$C_a^{Cl}(u,v)=\max\left[\left(u^{-a}+v^{-a}-1\right)^{-\frac{1}{a}},0\right], \tag{11}$$

and Frank copula function (Frank, 1979),

$$C_a^{Fr}(u,v)=-\frac{1}{a}\ln\left[1+\frac{\left(e^{-au}-1\right)\left(e^{-av}-1\right)}{e^{-a}-1}\right]. \tag{12}$$

2.2.2 Parameters estimation

There exist three main approaches to parameter estimation for copula function based dependency modelling: exact maximum likelihood method (EMLM), inference function for margins (IFM), and canonical maximum likelihood (CML). While for the former all parameters are estimated within one step, which might be very time consuming (mainly for high dimensional problems or complicated marginal distributions), the latter two methods are based on the estimation of the parameters for the marginal distribution and parameters for the copula function separately. While assuming IFM, marginal distributions are estimated in the first step and the copula function in the second one, for CML instead of parametric margins empirical distributions are used. In this paper we apply IFM estimation method.

3. Backtesting procedure

Within the backtesting procedure, the ability of a given model to estimate the future losses is tested. Backtesting is based on the estimation of the risk (mostly measured as Value at Risk) at time t

for time $t + \Delta t$, where Δt is usually (in line with the standards for bank supervision as defined within Basel II) set to one business day, and comparison with the true loss observed at time $t + \Delta t$. This procedure is applied for moving time window over the whole utilized data set.

Within the backtesting procedure on a given time series the following two situations can arise – the loss is higher or lower than its estimation. While the former case is denoted by 1 as an exception, the latter one is denoted by zero. For further details see Hull (2006) or Resti and Sironi (2007). In this way, we obtain the sequence of logical values corresponding to the fact whether the exception has occurred or not. We get the sequence I_t,

$$I_t = \begin{cases} 1 & if\ r_t < -VaR_t \\ 0 & if\ r_t \geq -VaR_t \end{cases}. \tag{13}$$

On this sequence it can be tested, whether the number of ones (exceptions) corresponds with the assumption, i.e. $(1-\alpha)\cdot n$ (where n is the length of the data set), whether the estimation is valid either unconditionally or conditionally, whether bunching is present, etc. In this paper we define only unconditional coverage test proposed by Kupiec (1995) and conditional coverage test due to Christoffersen (1998).

Kupiec's test (henceforth K-test) is derived from a relative amount of exceptions, i.e. whether their quantity is from the statistical point of view different from the assumption. The null hypothesis is that the observed probability of exception occurring is equal to the assumed. A given likelihood ratio on the basis of χ^2 probability distribution with one degree of freedom is formulated as follows:

$$LR = -2\ln\left[\frac{\pi_{ex}^{n_1}\left(1-\pi_{ex}\right)^{n_0}}{\pi_{obs}^{n_1}\left(1-\pi_{obs}\right)^{n_0}}\right], \tag{14}$$

where π_{ex} is expected probability of exception occurring, π_{obs} is observed probability of exception occurring, n_0 is the number of zeros and n_1 is the number of ones (exceptions). The Kupiec's test takes into account only the quantity of exceptions.

By contrast, in order to assess whether the exceptions are distributed equally in time, i.e. without any dependence (autocorrelation), we should define the time lag first: in Christoffersen (1998) it is defined as the stage, when exception at one moment in time can significantly help to identify whether another exception will (not) follow on the subsequent day. Therefore, we replace the original sequence by a new one, where 01, 00, 11 or 10 is recorded. The null hypothesis is that the probability of exception occurring is independent on the information whether the exception has occurred also previous day. Then we have the likelihood ratio as follows (henceforth C1-test):

$$LR = -2\ln\left[\frac{\pi_{obs}^{n_1}\left(1-\pi_{obs}\right)^{n_0}}{\pi_{01}^{n_{01}}\left(1-\pi_{01}\right)^{n_{00}}\pi_{11}^{n_{11}}\left(1-\pi_{11}\right)^{n_{10}}}\right], \tag{15}$$

where $\pi_{ij} = \Pr\left(I_t = j \middle| I_{t-1} = i\right)$ and $\pi_{obs} = \dfrac{n_{01}+n_{11}}{n_{00}+n_{01}+n_{10}+n_{11}}$. This test statistic has χ^2 probability distribution with one degree of freedom.

Obviously, we can evaluate these two tests together by calculating the following likelihood ratio (henceforth C2-test):

$$LR = -2\ln\left[\frac{\pi_{ex}^{n_1}\left(1-\pi_{ex}\right)^{n_0}}{\pi_{01}^{n_{01}}\left(1-\pi_{01}\right)^{n_{00}}\pi_{11}^{n_{11}}\left(1-\pi_{11}\right)^{n_{10}}}\right], \tag{16}$$

which has χ^2 probability distribution with two degrees of freedom.

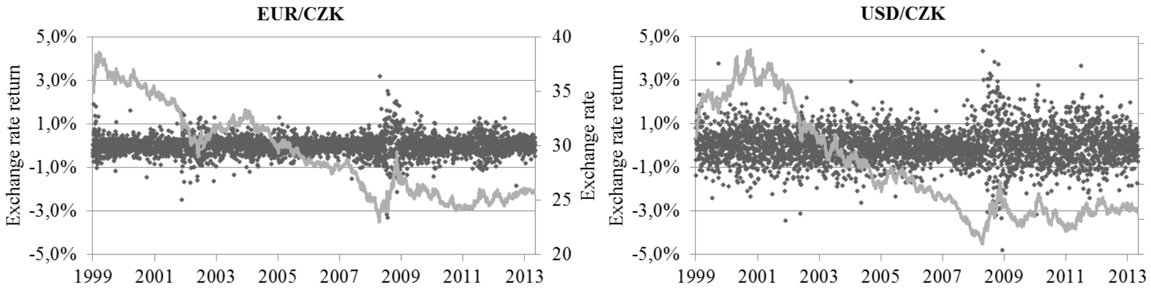
4. Application part

In this paper we try to examine whether the GARCH models with residuals joined by a suitable copula function are eligible to estimate the currency risk of foreign exchange rate sensitive portfolio. For application part, the simple portfolio sensitive to EUR/CZK foreign exchange pair (long) and USD/CZK foreign exchange pair (short) is assumed. Due to this simple composition of portfolio, the results obtained can be compared to the results when assuming portfolio sensitive to one risk factor only – EUR/USD foreign exchange pair (long). Besides data description (subsection 4.1) the section is organised in accordance with the modelling procedure, i.e. interesting results concerning the GARCH model estimations (subsection 4.2) and copula functions estimations (subsection 4.3) are presented first. Then there are presented the backtesting results of portfolio risk estimation (subsection 4.4), which is done as the single risk driver modelling (EUR/USD foreign exchange pair) and two risk drivers modelling (EUR/CZK and USD/CZK foreign exchange pairs).

4.1 Dataset description

For the backtesting purposes the exchange rates of EUR/CZK and USD/CZK currency pairs were assumed. The data were downloaded from the webserver of Czech National Bank.[1] The length of the time series is 3,472 daily exchange rates covering the period from January 1, 1999 till September 30, 2013. From these data the continuous daily returns were calculated.[2] The evolutions of the exchange rates as well as the daily returns are depicted in Figure 1.

Figure 1: Exchange rate evolutions and corresponding daily returns



Source: Czech National Bank

From the figure it can be seen that the Czech crown was steadily appreciating during the examined period. The only sharp depreciation of the Czech crown can be observed during the second half of the year 2008. This depreciation is connected to the financial crisis and to the increase of risk aversion which caused that the investors wanted to have their money denominated in safe currencies (such as USD or EUR). From the figure it is also clear that the returns of USD/CZK are more volatile compared to EUR/CZK.

The basic descriptive statistics of the returns such as the minimum, maximum, mean and median, standard deviation, skewness and kurtosis are summarized in Table 1. From the table the overall decrease in (both) exchange rates can be confirmed (both mean and median are lower than zero). We can also see that there is almost double volatility of USD/CZK compared to the EUR/USD (see the standard deviation). However, EUR/USD possess more extreme returns than USD/CZK (due to the almost double value of kurtosis). Both distributions of returns are positively skewed as both the mean is bigger than median and skewness is positive. This means that we can observe more losses (but smaller) than gains.

Table 1: Basic descriptive characteristics of exchange rates returns (daily data)

Descriptive characteristics	EUR/CZK	USD/CZK	EUR/USD
Minimum	-3.316%	-5.737%	-4.821
Maximum	3.186%	4.333%	4.056
Mean	-0.008%	-0.012%	0.004%
Median	-0.015%	-0.026%	0.017%

[1] http://www.cnb.cz/cs/financni_trhy/devizovy_trh/kurzy_devizoveho_trhu/rok_form.jsp
[2] Both for these foreign exchange rates and composite EUR/USD foreign exchange rate.

standard deviation	0.405%	0.790%	0.656%
Skewness	0.164	0.065	-0.043
Kurtosis	10.339	5.834	5.235

Source: author's calculations

In line with the described backtesting procedure (section 3) we estimate the GARCH-copula model on rolling window of 250 days (i.e. approximately one year), i.e. first 250 observations are left for the parameters estimation for day 251. For next days the estimation procedure is repeated 3,221 times, in each step the parameters are estimated from preceding 250 observations prior to the examined day. Value at Risk of portfolio stated by the estimated model is in each step compared to the (true) observed loss and the information about (not) occurring of exception is obtained. For simplicity we assume the value of the portfolio equal to 1, thus Value at Risk can be directly compared to the return of EUR/USD foreign exchange pair.

4.2 GARCH model estimations

The parameters of GARCH model were estimated in MATLAB by means of method of maximum likelihood by applying function *garchfit*. The values of *R*, *P*, *Q* were chosen in line with the described procedure (section 2.1), the quantities of different values for each foreign exchange pair are depicted in Table 2. For the simplicity the models are labelled in accordance to their structure as *C-R-P-Q*. In this label, *C* stands for the constant μ_0 (1 if it is present, 0 if statistically insignificant), *R*, *P*, *Q* are the value of lags as specified above in subsection 2.1.

Table 2: The quantities of different models' structure utilization

Model structure	EUR/CZK	USD/CZK	EUR/USD	Total
0-0-0-0	1259 (39.1%)	2099 (65.2%)	1868 (58%)	5226 (54.1%)
0-0-0-1	170 (5.3%)	15 (0.5%)	239 (7.4%)	424 (4.4%)
0-0-1-1	1168 (36.3%)	563 (17.5%)	905 (28.1%)	2636 (27.3%)
1-0-1-1	310 (9.6%)	218 (6.8%)	150 (4.7%)	678 (7%)
1-1-1-1	244 (7.6%)	78 (2.4%)	59 (1.8%)	381 (3.9%)
1-1-2-1	24 (0.7%)	0 (0%)	0 (0%)	24 (0.2%)
1-2-1-1	0 (0%)	71 (2.2%)	0 (0%)	71 (0.7%)
1-2-2-1	45 (1.4%)	177 (5.5%)	0 (0%)	222 (2.3%)
1-2-2-2	1 (0%)	0 (0%)	0 (0%)	1 (0%)

Source: author's calculations

From Table 2 we can see that in the most rolling windows all the parameters were found statistically insignificant (i.e. the time series are modelled as a Wiener motion with Student distribution instead of Gaussian). Assuming GARCH model, only parameters of lag one were found statistically significant with insignificant level constant in returns equation the most frequently (on average 27.3% of rolling windows).

Generally we can conclude, that the simpler GARCH model (structure 1-1) were applied more frequently, if it was applied at all. Also, auto-regression coefficients of returns and return constant were mostly found insignificant.

4.3 Copula function estimations

The standardized residuals obtained by the GARCH filtering were used for copula parameter estimations. First the Student distribution (of these standardized residuals) were changed to uniform distribution applying estimated cumulative distribution function (function *cdf* in MATLAB) and then the copula function parameters were estimated by means of method of maximum likelihood (function *copulafit* in MATLAB) in accordance with the description of IFM estimation method (section 2.2.2).

The suitability of assumed copula functions was compared based on the likelihood function. The rank quantities are summarized in Table 3.

Table 3: Rank summarizations (quantities) of different copula functions

Rank	Gaussian	Student	Clayton	Frank	Gumbel
The best	0 (0%)	2641 (82%)	66 (2%)	193 (6%)	321 (10%)
Second best	1118 (35%)	549 (17%)	354 (11%)	103 (3%)	1097 (34%)
Third best	1399 (43%)	31 (1%)	364 (11%)	845 (26%)	582 (18%)
Fourth best	686 (21%)	0 (0%)	829 (26%)	978 (30%)	728 (23%)
The worst	18 (1%)	0 (0%)	1608 (50%)	1102 (34%)	493 (15%)

Source: author's calculations

From the table we can see, that the most frequently chosen copula function was the Student copula (it was chosen in 82% rolling windows). The appropriate comparison to the Gaussian copula function can be made – it can be seen that the Gaussian copula function was never chosen as the best. This is because Student copula function was always superior to the Gaussian one; actually Gaussian copula is just the special case of Student copula when value of degrees of freedom approaches infinity. Thus Student copula should always fit the data better (respectively equally in extreme case). Concerning Archimedean copula functions, we can see that Clayton and Frank does not fit the data well (50% and 34% of time chosen as the worst), while the fitness of Gumbel copula is quite good (10% of time chosen as the best, 34% of time chosen as the second best), it is more or less comparable to the Gaussian.

4.4 Backtesting results

In the previous two subchapters some interesting findings concerning estimations of both GARCH process and copula function were presented. When all the necessary parameters were estimated the one day ahead portfolio return was simulated in each backtesting step using Monte Carlo simulation method and Value at Risk was calculated from theses simulations subsequently. Each backtesting step 100,000 trials were simulated and Value at Risk was calculated in MATLAB using function *quantile*. In line with the backtesting procedure (section 3) the observed numbers of exceptions were recorded (recall: exception is the case in which the true observed loss is higher than estimated Value at Risk), and P-values of K-test and C-tests were calculated. These computations were done in two ways: (i) assuming both risk factors (EUR/CZK and USD/CZK) and encompassing the dependency between them, and (ii) assuming only one risk factor (EUR/USD) and thus avoiding the dependency modelling issue – this approach is then viewed as the benchmark to the first approach.

4.4.1 Backtesting results of GARCH model for EUR/USD risk driver

Results obtained by backtesting of one risk factor only modelling are summarised in Table 4. As can be seen, although there were observed slightly more exceptions than assumed, the quantity of exceptions and thus also model accuracy can be statistically accepted (for all probability levels α). Also bunching of exceptions is not a problem, as all the p-values of both C-tests were found higher than 5%, although from table 2 we know that in majority of rolling windows (58% of cases) only simple model of Student distribution without conditional variance were applied. Generally, we can conclude that modelling of EUR/USD can be done accurately by means of GARCH model and Student distribution, in more than half cases only Student distribution is enough.

Table 4: Backtesting results of GARCH model on EUR/USD exchange rate

Characteristics	$\alpha = 15\%$	$\alpha = 10\%$	$\alpha = 5\%$	$\alpha = 1\%$	$\alpha = 0.5\%$	$\alpha = 0.1\%$	$\alpha = 0.03\%$
# assumed	483.15	322.1	161.05	32.21	16.105	3.221	0.9663
# observed	501	350	180	40	22	6	2
K1 test	38.08%	10.56%	13.24%	18.37%	16.31%	16.71%	35.86%
C1 test	42.10%	11.31%	52.86%	52.43%	58.22%	88.10%	96.02%
C2 test	48.95%	7.62%	26.26%	33.66%	32.44%	38.03%	65.52%

Source: author's calculations

4.4.2 Backtesting results of GARCH -copula model

Assuming two risk factors and applying proposed GACRH-copula model we obtained backtesting results, which are summarized in Table 5. For the probability levels of 15% and 10% the quantity of exceptions, i.e. the accuracy of model, improved (484 exceptions compared to 501 and 331 compared to 350). However the bunching of exceptions emerged, as their dependence in lag one was confirmed by means of C1-test. For probability level of 5% both the accuracy of model and bunching of exceptions deteriorated, however the accuracy is still acceptable (p-value of K-test is higher than 5%). For lower probability levels the accuracy of the model deteriorated even so much that the model cannot be statistically accepted. Nevertheless, the bunching of exceptions was not detected by C1-test.

Table 5: Backtesting results of GARCH-copula model (the best copula function is chosen in each step)

Characteristics	$\alpha = 15\%$	$\alpha = 10\%$	$\alpha = 5\%$	$\alpha = 1\%$	$\alpha = 0.5\%$	$\alpha = 0.1\%$	$\alpha = 0.03\%$
# assumed	483.15	322.1	161.05	32.21	16.105	3.221	0.9663
# observed	484	331	182	51	34	14	10
K1 test	96.66%	60.26%	9.68%	0.22%	0.01%	0.00%	0.00%
C1 test	2.02%	0.65%	4.04%	83.39%	39.43%	72.66%	80.29%
C2 test	6.73%	2.14%	3.06%	0.88%	0.04%	0.01%	0.00%

Source: author's calculations

In our backtesting procedure we changed the structure of the model in each backtesting step in dependence on the most appropriate GARCH structure and type of copula function. It can be also interesting to investigate how the results would deteriorate[3] if we fix either GARCH structure or copula function. Thus, model was backtested once more fixing the choice of copula function. The results for different copula functions are summarised in Table 6 in the appendix of the paper. As one would expect, fixing the type of copula function would not improve the results for all the probability levels. Nevertheless, applying Frank copula function the results improve for all the probability levels except the 15%, although this does not change the conclusions about the model accuracy for particular probability levels. Next, if we fix the copula type to Clayton, the accuracy of model deteriorate in probability levels of 15% and 10% (the quantity of exceptions is now too low for model to be accepted), however for 1% and 0.5% levels the accuracy improves and model can then be statistically accepted as accurate. The same can be investigated by fixing GARCH structure. This issue will be the subject for further research.

5. Conclusion

Modelling of financial time series is clearly difficult but not less important part of financial risk management. The difficulties of modelling are caused by the specific characteristics of financial time series, such as fat tails, volatility clustering and dependence, which cannot be easily modelled. In the paper simple GARCH-copula methodology were proposed and backtested on a simple portfolio

[3] Actually, the deterioration of the backtesting results is only assumed – by assuming only one type of copula function or GARCH structure the possibility to fit the data are decreased, however the probability of overfitting decreases as well.

sensitive to the two foreign exchange pairs (two risk factors). The intentional choice of sensitivity (long/short positions) and concrete foreign exchange pairs allowed us to model the portfolio risk by two ways: (i) as a portfolio sensitive to only one composite risk driver (EUR/USD) and (ii) as a portfolio sensitive to both risk drivers (EUR/CZK and USD/CZK) with the necessity to model the mutual dependence. The first way of modelling can be viewed as a benchmark for the second way.

It was found out, that while the portfolio risk can be accurately modelled by the first way, the accuracy of results deteriorated for probability levels lower than 5% when assuming the dependency. From the examination of utilized GARCH model structures it was found out that in the most cases (54% of them) all the parameters in conditional volatility equation were found insignificant and in almost one fourth of cases the GARCH model with only lag one parameters were found significant.

As the accuracy of proposed GARCH-copula model were worse compared to the model neglecting the dependency modelling, there are obviously areas for further researches. Other copula functions, or more preferably their combinations can be assumed. Empirical marginals/copula can also be applied. Another research area is to backtest different measures of risk. In the paper the Value at Risk was applied as a measure of the risk, further research can be done applying conditional Value at Risk measure.

Acknowledgement

The research was supported by the European Social Fund under the Opportunity for young researchers project (CZ.1.07/2.3.00/30.0016) as well as by GAČR (Czech Science Foundation – Grantová Agentura České Republiky) under the project no. 13-18300P.

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Appendix

Table 6: Backtesting results of GARCH-copula model with different copula functions utilized

Copula function	Characteristics	$\alpha =$ 15%	$\alpha =$ 10%	$\alpha = 5\%$	$\alpha =$ 1%	$\alpha =$ 0.5%	$\alpha =$ 0.1%	$\alpha =$ 0.03%
Independent	# observed	290	172	68	16	13	6	3
	K1 test	0.0%	0.0%	0.0%	0.1%	42.2%	16.7%	9.8%
	C1 test	30.5%	11.6%	64.9%	68.9%	74.5%	88.1%	94.0%
	C2 test	0.0%	0.0%	0.0%	0.6%	68.8%	38.0%	25.4%
Gaussian	# observed	447	298	164	51	33	16	12
	K1 test	7.1%	15.2%	81.2%	0.2%	0.0%	0.0%	0.0%
	C1 test	1.0%	3.7%	2.7%	20.0%	40.8%	68.9%	76.4%
	C2 test	0.7%	4.1%	8.5%	0.4%	0.1%	0.0%	0.0%
Student	# observed	492	340	187	49	30	11	6
	K1 test	66.3%	29.7%	4.1%	0.6%	0.2%	0.1%	0.1%
	C1 test	3.5%	4.6%	0.8%	21.8%	45.3%	78.4%	88.1%
	C2 test	9.8%	7.8%	0.3%	1.0%	0.6%	0.3%	0.3%
Clayton	# observed	434	282	138	37	22	13	6
	K1 test	1.4%	1.6%	5.6%	40.7%	16.3%	0.0%	0.1%
	C1 test	2.3%	5.0%	10.7%	35.4%	58.2%	74.5%	88.1%
	C2 test	0.4%	0.8%	4.5%	46.1%	32.4%	0.0%	0.3%

Table 6 (continuation): Backtesting results of GARCH-copula model with different copula functions utilized

Copula function	Characteristics	$\alpha =$ 15%	$\alpha =$ 10%	$\alpha =$ 5%	$\alpha =$ 1%	$\alpha =$ 0.5%	$\alpha =$ 0.1%	$\alpha =$ 0.03%
Frank	# observed	490	329	175	45	30	14	7
	K1 test	73.6%	68.6%	26.6%	3.3%	0.2%	0.0%	0.0%
	C1 test	2.0%	8.1%	0.4%	25.9%	45.3%	72.7%	86.1%
	C2 test	6.3%	20.1%	0.8%	5.4%	0.6%	0.0%	0.0%
Gumbel	# observed	492	346	193	53	34	15	8
	K1 test	66.3%	16.5%	1.2%	0.1%	0.0%	0.0%	0.0%
	C1 test	4.8%	3.6%	0.7%	89.2%	39.4%	70.8%	84.2%
	C2 test	12.8%	4.2%	0.1%	0.3%	0.0%	0.0%	0.0%
# assumed		483.15	322.10	161.05	32.21	16.11	3.22	0.97